

AMERICAN SOCIETY OF CIVIL ENGINEERS.

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No. 859.

GENERAL CRITERION FOR POSITION OF LOADS
CAUSING MAXIMUM STRESS IN ANY
MEMBER OF A BRIDGE TRUSS.

By L. M. HOSKINS, Esq.

PRESENTED APRIL 5TH, 1899.

WITH DISCUSSION.

The most dangerous load to which a railway bridge is subjected consists of a somewhat irregularly distributed series of train weights, applied to the track at the points of contact of the wheels, and to the trusses at the points of support of the track system. The actual distribution of such a load among the joints of the trusses is in some degree uncertain; some assumption must be made before the problem of computing stresses becomes determinate. In view of this uncertainty, some engineers prefer to simplify the problem by replacing the actual load by an "equivalent uniform-load," or by a series of equal panel loads, or by some combination of these. Others believe that a closer approximation to actual conditions is obtained by using a series of concentrated loads representing as nearly as possible the actual weights carried by the wheels of the locomotives and cars, and assuming each load to be applied to the truss as if carried by a simple beam supported at the two adjacent panel joints.

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It is no part of the object of this paper to consider the relative merits of these different assumptions. The results of the following discussion, although especially useful in treating the case of actual wheel loads, are wholly general, so far as the character of the load series is concerned. This paper is offered as a contribution toward the unification of the theory of maximum stresses due to moving loads.

This theory, with applications to the various cases of practical importance, is quite fully treated in manuals and text-books, and in special papers dealing with particular methods.* The problem presents no serious difficulties of principle, but the treatment usually given lacks generality. This is especially true of that part of the theory which deals with the criterion for position of loads causing maximum stress, a special analysis being commonly given for each of a number of particular cases. From this lack of generality arises the chief difficulty in applying the rules deduced. The use of a special rule or formula in each of a number of cases, each involving a special notation, is likely to prove confusing, even when each case in itself is sufficiently simple.

In the endeavor to generalize the treatment of this problem, the author was led to a result simple in form and easily applicable to the various particular cases. By the use of a simple general notation, the criterion for maximum stress assumes the same form for all members, whatever their position in the truss.† The deduction of this general criterion, with examples illustrating its application in various cases, is the object of this paper.

GENERILITY OF THE DISCUSSION.

It will be well to make clear at the outset the exact degree of generality of the discussion.

As regards form of truss and choice of member, there is no restric-

* "The Stresses in Framed Structures," by A. Jay Du Bois.

"The Theory and Practice of Modern Framed Structures," by J. B. Johnson, C. W. Bryan and F. E. Turneaure.

"Text Book on Roofs and Bridges," by Mansfield Merriman and H. S. Jacoby.

"On the Calculation of the Stresses in Bridges for the Actual Concentrated Loads," by George F. Swain. *Transactions, Am. Soc. C. E.*, Vol. xvii (1887).

"A New Graphical Solution of the Problem, What Position a Train of Concentrated Loads Must Have in Order to Cause the Greatest Stress in Any Given Part of a Bridge Truss or Girder," by Henry T. Eddy. *Transactions, Am. Soc. C. E.*, Vol. xxii (1890).

† The general formula expressing this criterion was given by the author in a paper presented to the Wisconsin Academy of Sciences, Arts and Letters, in December, 1893, and published in Vol. x of the *Transactions of the Academy*. The proof there given, though rigorous, was less simple than that now offered. A third proof, based upon a general graphical treatment of the problem, and depending upon elementary properties of the funicular polygon, is given in the author's "Elements of Graphic Statics," Revised Edition (1899).

tion, except that the stress must be "simply determinate." By this it is to be understood that the member under discussion may be taken as one of three, through which a section may be passed completely dividing the truss. Figs. 1 to 6 show six cases to which the analysis applies, the member under consideration being in every case marked k , and the section dividing the truss being represented by a broken line.

As regards the load series, this may consist of any distributed or concentrated loads. A concentrated load, however, is to be regarded as distributed over a finite, though short, horizontal length. This does not limit the discussion, so far as practical applicability is concerned, since the concentrated loads of practice do, in fact, act in the manner assumed. A concentrated load, in the strict mathematical sense (a finite load applied at a mathematical point), has only an ideal existence. Even such ideal loads may be regarded as limiting cases of distributed loads, and there is no difficulty in giving a discussion rigorously applicable to such an ideal load series. But the language used in the following analysis is exact only when such ideal concentrated loads are excluded.

DEFINITIONS AND NOTATION.

Stress-Moment and Moment-Center.—Suppose the stress in any member to be determined by the "method of moments." For this purpose the truss is conceived to be divided by a section cutting the member in question and two others; the intersection of the axial lines of these two is taken as the origin of moments in applying the equation of equilibrium to either of the two parts of the truss. Let this origin be called the moment-center for the member under consideration, and let the moment of the required stress, with respect to this origin, be called the stress-moment. Numerically, the value of the stress-moment is equal to the sum of the moments of the external forces acting upon either of the two portions into which the truss is divided by the section; algebraically, two cases are to be distinguished by the signs plus and minus. Calling clock-wise rotation negative, let the stress-moment be called positive when it resists a tendency of the left-hand portion of the truss to rotate negatively, and of the right-hand portion to rotate positively, with reference to the moment-center.

The kind of stress in any member may be determined by inspection

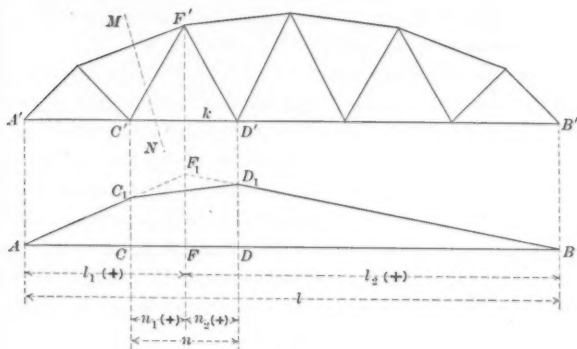


FIG. 1.

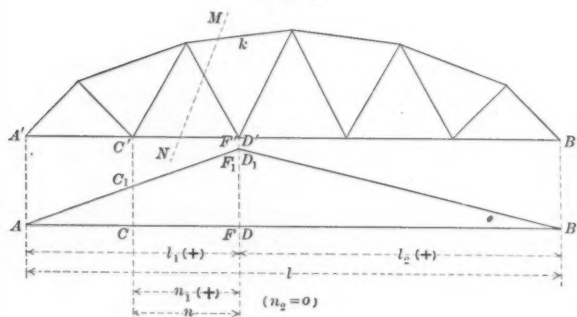


FIG. 2.

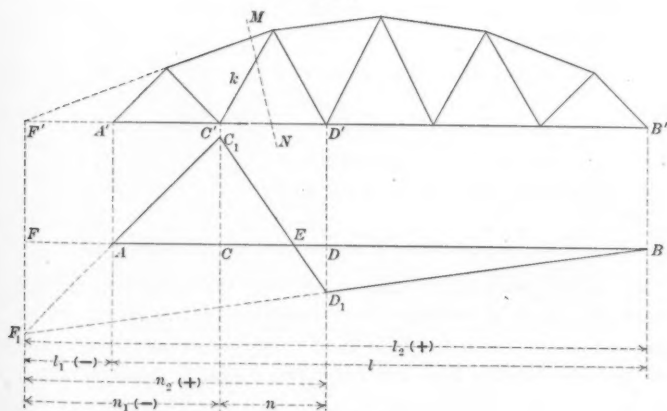


FIG. 3.

of the truss diagram, when the sign of the stress-moment and the position of the moment-center are known.

Notation.—Of the three members cut, in dividing the truss, one belongs to the chord at whose joints the moving loads are applied to the truss. Let the ends of this member be designated in all cases by C' and D' , and let A' and B' be the points of support of the truss. Let the moment-center for any chosen member be designated by F' . Projecting the five points A' , B' , C' , D' and F' upon a horizontal line, let their projections be marked A , B , C , D and F . The relative positions of these points in each of several cases are shown in Figs. 1 to 6, the member whose stress is under consideration being in every case marked k .

$$\begin{aligned} \text{Let } AB = l, AF = l_1, FB = l_2; \\ CD = n, CF = n_1, FD = n_2. \end{aligned}$$

Let the signs of the four quantities l , l_1 , n_1 and n_2 , conform to the order of the letters used in defining them, the positive direction being from left toward right. Thus l_1 and l_2 are both positive if F falls between A and B ; n_1 and n_2 are both positive if F falls between C and D . In all cases $l_1 + l_2 = l$ and $n_1 + n_2 = n$.

Let the load per unit length at any point be denoted by w . The only restriction as to the value of w is that it shall be either finite or zero at every point. The only case thus excluded is that of ideally concentrated loads above mentioned. At the point of application of an actual concentrated load w becomes very great, but not infinite.

Let the values of w at A , B , C and D , respectively, be denoted by w_a , w_b , w_c and w_d .

Let W = total load on the span AB ;

Q = " " " panel CD ;

P_1 = " " " AC ;

P_2 = " " " DB .

Identically, $W = P_1 + P_2 + Q$.

DEDUCTION OF CRITERION FOR MAXIMUM OR MINIMUM STRESS.

In the following analysis, direct reference will be made to the case shown in Fig. 1, in which the four quantities l , l_1 , n_1 and n_2 are all positive. It will be seen, however, that the reasoning applies to any of the cases shown in Figs. 2 to 6 if regard be had for algebraic signs.

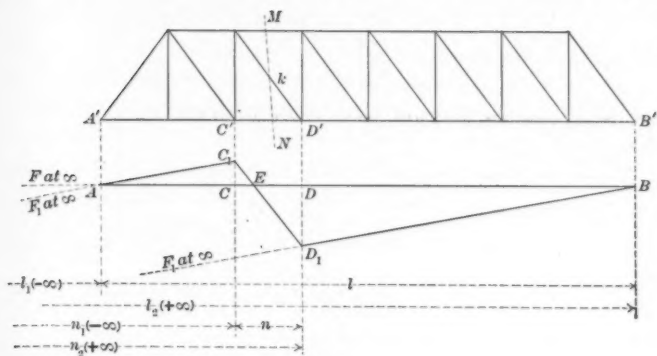


FIG. 4.

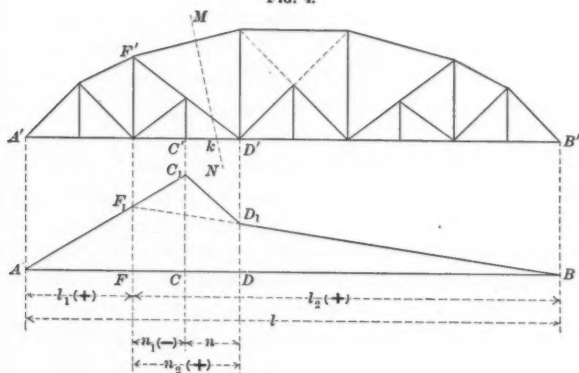


FIG. 5.

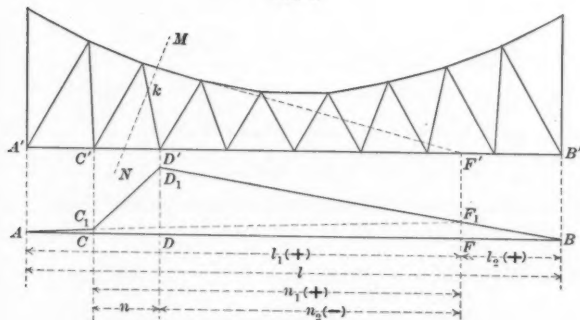


FIG. 6.

Value of Stress-Moment.—The magnitude of the resultant load on AC is P_1 ; let the distance of its line of action from A be p_1 . Similarly, let P_2 , the resultant of the loads on DB , act in a line distant p_2 from B ; and let Q , the resultant of the loads on CD , act in a line distant q_1 from C and q_2 from D .

The stress-moment due to P_1 may be found by applying the principles of equilibrium to the portion of the truss at the right of the section MN (Fig. 1). The reaction at B is $\frac{P_1 p_1}{l}$; the moment of this reaction with respect to the moment-center is $\frac{l_2}{l} (P_1 p_1)$, which is therefore the value of the stress-moment due to P_1 .

The stress-moment due to P_2 is found in a similar manner, by considering the equilibrium of the portion of the truss to the left of the section MN ; the value being $\frac{l_1}{l} (P_2 p_2)$.

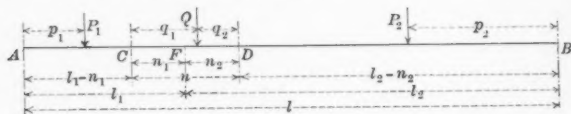


FIG. 7.

Of the load Q , the portion $\frac{Qq_2}{n}$ is applied to the truss at C , and the portion $\frac{Qq_1}{n}$ at D . The effect of the former is found as in the case of P_1 ; $l_1 - n_1$ taking the place of p_1 and $\frac{Qq_2}{n}$, the place of P_1 ; the stress-moment due to this part of Q is therefore $\frac{(l_1 - n_1) l_2}{n l} (Qq_2)$; while the stress-moment due to the portion at D , determined as in the case of P_2 , is in like manner found to have the value $\frac{(l_2 - n_2) l_1}{n l} (Qq_1)$. These values are all positive in the case shown in Fig. 1.

The stress-moment due to the total load on the truss is the sum of these four quantities, or

$$M = \frac{l_2}{l} (P_1 p_1) + \frac{l_1}{l} (P_2 p_2) + \frac{(l_2 - n_2) l_1}{n l} (Q q_1) + \frac{(l_1 - n_1) l_2}{n l} (Q q_2) \dots \dots \dots (1)$$

Change of Stress-Moment Due to Small Displacement of Loads.—Let the load-series receive a small displacement δx toward the left (this being taken as the positive direction for displacements), and let δM denote the resulting increment in the value of M . To compute δM it is necessary to determine the increments of the four quantities $P_1 p_1$, $P_2 p_2$, $Q q_1$ and $Q q_2$. Of these quantities, the first is the moment of P_1 with respect to A , the second is the moment of P_2 with respect to B , the third is the moment of Q with respect to C , and the fourth is the moment of Q with respect to D .

During the displacement, loads $w_a \delta x$, $w_b \delta x$, $w_c \delta x$ and $w_d \delta x$ pass the points A , B , C and D , respectively.

After the displacement, the load which has replaced P_1 is the resultant of three components, as follows: (1) The original load P_1 , acting in a line distant $p_1 - \delta x$ from A , (2) a load $w_c \delta x$, acting just at the left of C , this being the amount of load which passes C during the displacement, and (3) a negative load equal to $w_a \delta x$, acting just at the left of A , this being the amount of the load which passes off the truss at A during the displacement. The new value of $P_1 p_1$ is the sum of the moments of these three components with respect to A . Therefore, neglecting powers of δx higher than the first, we have, as the value of $P_1 p_1$, after the displacement, $P_1 (p_1 - \delta x) + w_c (l_1 - n_1) \delta x$.

Similarly, the load which has replaced P_2 is the resultant of a load P_2 acting at a distance $p_2 + \delta x$ from B , a load $w_b \delta x$ acting just at the left of B , and a negative load $w_d \delta x$ acting just at the left of D . The new value of $P_2 p_2$ is the sum of the moments of these three components about D , or $P_2 (p_2 + \delta x) - w_d (l_2 - n_2) \delta x$.

Similar reasoning gives for the new values of $Q q_1$ and $Q q_2$, $Q (q_1 - \delta x) + w_a n \delta x$ and $Q (q_2 + \delta x) - w_c n \delta x$.

The value of M after the displacement is, therefore,

$$M + \delta M = \frac{l_2}{l} \left[P_1 (p_1 - \delta x) \right] + \frac{l_1}{l} \left[P_2 (p_2 + \delta x) \right] \\ + \frac{(l_2 - n_2) l_1}{n l} \left[Q (q_1 - \delta x) \right] + \frac{(l_1 - n_1) l_2}{n l} \left[Q (q_2 + \delta x) \right]$$

the terms involving w_c and w_d disappearing from the simplified result.

The increment of M caused by the displacement is

$$\delta M = \left(-\frac{l_2}{l} P_1 + \frac{l_1}{l} P_2 + \frac{l_1 n_2 - l_2 n_1}{l n} Q \right) \delta x.$$

Dividing by δx and passing to the limit, δx approaching zero,

$$\frac{dM}{dx} = \frac{l_1}{l} (P_2 + \frac{n_2}{n} Q) - \frac{l_2}{l} (P_1 + \frac{n_1}{n} Q) \dots \dots \dots (2)$$

Condition for Maximum or Minimum Stress.—The condition for maximum or minimum value of M (and therefore also of the stress), is $\frac{dM}{dx} = 0$; which gives the equation

$$\frac{P_1 + \frac{n_1}{n} Q}{l_1} = \frac{P_2 + \frac{n_2}{n} Q}{l_2} = \frac{W}{l} \dots \dots \dots (3)$$

the third member of the equation being obtained by adding the numerators and denominators, respectively, of the fractions forming the first and second members.

This general equation is satisfied whenever the stress in the member is a maximum or a minimum. That it applies to any one of the cases shown in Figs. 2, 3, 4, 5 and 6, may be verified by following the above reasoning through, taking account of the algebraic sign of every quantity dealt with.*

Special Forms of the General Criterion.—For practical use it will generally be found convenient to put Equation (3) in one of the following forms:

$$P_1 + \frac{n_1}{n} Q = \frac{l_1}{l} W \dots \dots \dots (3a)$$

$$P_2 + \frac{n_2}{n} Q = \frac{l_2}{l} W \dots \dots \dots (3b)$$

unless the moment-center is at infinity. In this case (Fig. 4) these equations take indeterminate forms. But since when this occurs $\frac{l_1}{n_1} = 1$ and $\frac{l_2}{n_2} = 1$, both equations may be reduced to the form:†

$$Q = \frac{n}{l} W \dots \dots \dots (3c)$$

In addition to the general criterion expressed by Equation (3), it is necessary to have some means of distinguishing between positions giving maximum and those giving minimum values of the stress. Further, since the stress in any member will, in general, pass through a number

* For the case represented in Fig. 1, in which the member under consideration belongs to the loaded chord, l_1 , l_2 , n_1 and n_2 being all positive, formulas identical with Equation (3) have been given by several writers. The generality of the formula seems not to have been recognized.

† Equation (3c) is identical with the condition for maximum shear in the panel containing the web member under consideration. It is, of course, only when the chords are parallel that the maximum shear and maximum stress can be assumed to occur simultaneously.

of maximum values as the loads pass over the bridge, means are needed for selecting the greatest of the possible maximum values.

A general criterion for distinguishing between positions giving maximum and those giving minimum values will be deduced below. It will be well, however, first to consider the question in a somewhat less general manner. The problem of distinguishing between maxima and minima, as well as that of selecting the greatest from the possible maximum values, may be treated advantageously by the use of "influence diagrams."

INFLUENCE DIAGRAMS.

A very full and clear discussion of influence lines, with applications to the determination of maximum stresses in various cases of practical importance, has been given by George F. Swain, M. Am. Soc. C. E., in the paper cited on page 241. It will be advantageous to present here a brief treatment of this subject from the general point of view adopted in the foregoing analysis.

Influence Line.—The influence line, for a given member of a truss, is a line whose ordinate at any point represents the value of the stress in the member due to a unit load at that point.

General Method of Drawing Influence Line.—In the following discussion any ordinate of the influence line will be regarded as agreeing in sign with the corresponding value of the stress-moment, according to the convention already explained. Positive ordinates will be drawn upward from the reference line, negative ordinates downward.

If M denotes the stress-moment due to a unit load in a given position, and t the perpendicular distance from the moment-center to the axial line of the member under consideration, $\frac{M}{t}$ denotes the value of the ordinate of the influence line.

Referring first to the case in which l_1 , l_2 , n_1 and n_2 are all positive (Fig. 1), and repeating the reasoning already given in determining values of the stress-moment due to P_1 , P_2 and Q , the following results are obtained:

A unit load between A and C , distant x from C , causes a stress $\frac{l_2 x}{lt}$. This changes from 0 to $\frac{l_2 (l_1 - n_1)}{lt}$ as the load passes from A to C .

Similarly, as a unit load passes from B to D , the stress due to it changes from 0 to $\frac{l_1(l_2 - n_2)}{lt}$.

Of a unit load on CD , distant x from C , the portion $\frac{n-x}{n}$ comes upon the truss at C and $\frac{x}{n}$ at D . The former causes the stress $\frac{l_2(l_1 - n_1)}{lt} \times \frac{n-x}{n}$, and the latter the stress $\frac{l_1(l_2 - n_2)}{lt} \times \frac{x}{n}$. As the load passes from C to D , the sum of these values changes from $\frac{l_2(l_1 - n_1)}{lt}$ to $\frac{l_1(l_2 - n_2)}{lt}$.

From these results it is seen that the influence line consists of three straight lines AC_1 , C_1D_1 , D_1B ; the ordinates, CC_1 and DD_1 , having values:

$$CC_1 = \frac{l_2(l_1 - n_1)}{lt}; \quad DD_1 = \frac{l_1(l_2 - n_2)}{lt}.$$

If the various cases shown in Figs. 1 to 6 be considered, these values are found to be correct in magnitude and sign for all cases. The following general property may also readily be verified:

The lines AC_1 and BD_1 (produced if necessary) have a common ordinate at the point F , its value being $\frac{l_1 l_2}{lt}$.

This property gives the following general rule for drawing the influence line:

At F erect an ordinate equal in magnitude and sign to $\frac{l_1 l_2}{lt}$, and draw straight lines from its extremity to the points A and B . Connecting C_1 and D_1 , the points in which these lines intersect ordinates erected at C and D , respectively, the line AC_1D_1B is the influence line.

It will be seen that, so long as the moment-center falls within the span, so that both l_1 and l_2 are positive, the ordinates of the influence line are everywhere positive. This is the case in Figs. 1, 2, 5 and 6. There are, however, two types of influence line with ordinates everywhere positive, corresponding to the cases in which F falls within the panel CD (Fig. 1), and without the panel (Figs. 5 and 6). The limiting case between these two is shown in Fig. 2.

If the moment-center falls without the span, making l_1 or l_2 negative, the ordinates of the influence line are positive in one portion of

the span and negative in the remaining portion. Fig. 3 represents the case in which l_1 is negative, the ordinate CC_1 being positive and DD_1 negative. The line $C_1 D_1$ crosses AB at a point E , between C and D .

If the point F falls at infinity, as in Fig. 4, the above general rule for drawing the influence line by means of the ordinate at F becomes inapplicable. The lines $F_1 A$, $F_1 B$ are in this case parallel, and may be drawn by means of the general values previously given for the ordinates at C and D . It may be noticed also that if AC_1 be prolonged, its ordinate at B has the value $\frac{l_2}{t}$; and if BD_1 be prolonged, its ordinate at A has the value $-\frac{l_1}{t}$. These values are determinate even when l_1 , l_2 and t are infinite.

When the load series consists of concentrated loads, the question whether any certain position, in which the general equation (3) is satisfied, gives a maximum or a minimum value of the stress may usually be determined by simple inspection of the influence diagram. It is also often possible to determine, with high probability, which of several maximum values is greatest, without actually determining the values. These questions are now to be considered.

DISCRIMINATION BETWEEN MAXIMUM AND MINIMUM VALUES.

Application of Influence Diagram.—Referring to Fig. 1, and observing the slope of each of the three straight lines forming the influence diagram, consider how the stress in the member k , due to a single load, varies as the load moves uniformly from B toward the left. During the motion from B to D the stress increases at a uniform rate; from D to C it decreases at a uniform rate; from C to A it decreases uniformly at a greater rate. If the load moves from A to B , these effects are reversed.

Next consider the effect of displacing a series of loads. As above, let the total loads on AC , CD and DB , respectively, be denoted by P_1 , Q and P_2 . If the loads are displaced toward the left, the stress due to P_2 increases, while the stress due to P_1 , and also that due to Q , decreases. The resultant of these three effects may be either an increase or a decrease; but the resultant rate of change of the stress (whether increase or decrease) remains constant as long as P_1 , P_2 and Q remain constant, that is, as long as no load passes any one of the four points A , C , D and B .

As the loads approach a position causing a maximum stress, the value of the stress increases; as they recede from such a position, the value of the stress decreases. The reverse changes take place as the loads approach and recede from a position of minimum stress.

First, suppose the stress is increasing as the loads move toward the left. If a load comes on the truss at *B*, the rate of increase of the stress will become greater. The same will be true if a load passes off at *A*. If a load passes *C* or *D*, the rate of increase of the stress becomes less, and this effect may be sufficiently great to change the increase into a decrease; if so, the stress passes through a maximum value. Hence, only when a load is at *C* or at *D* can the stress be a maximum.

Next, suppose the stress is decreasing as the loads move toward the left. If a load passes *C* or *D*, the rate of decrease becomes greater. If a load comes on the truss at *B*, or if one leaves the truss at *A*, the rate of decrease becomes less, and this effect may be so great as to change the decrease into an increase; if so, the stress passes through a minimum value. Therefore, only when a load is at *A* or at *B* can the stress be a minimum.

Like reasoning obviously applies to the cases represented in Figs. 2, 3, 4, 5 and 6.*

Thus, in Fig. 2, a maximum can occur only with a load at *D*. A load at *C* can give neither a maximum nor a minimum, since *A C*, and *C₁ D₁* form a continuous straight line.

In the case of a web member, represented in Fig. 3, the stress may be either tension or compression. The kind of stress corresponding to a positive stress-moment (tension in this case), can have a maximum value with a load at *C* or at *B*, a minimum value with a load at *A* or at *D*. For the opposite kind of stress, these statements must be reversed. Like statements hold for the case in which *F* is at infinity (Fig. 4).

In Fig. 5, a maximum value of the stress in the member *k* can occur only when a load is at *C*; a minimum value may occur when a load is at *A*, at *D* or at *B*.

In Fig. 6, a maximum value can occur with a load at *D*, a minimum value with a load at *A*, at *C* or at *B*. This form of truss, though not of practical importance, is shown in order to illustrate the generality of the method. Its similarity to the case shown in Fig. 5 is apparent from

*The actual value of the rate of increase of the stress as the loads move may be computed from the slope of the three portions of the influence line. Using the values of the ordinates at *C* and *D*, as above given, this method leads easily to Equation (2), and thus also to the general criterion (3).

a comparison of the influence diagrams. It may be remarked that, as a special case of Fig. 6, the point F may coincide with B (or with A), making l_2 (or l_1) zero. The ordinate $F F_1$ would be zero, and the influence line would have to be constructed from the values of the ordinates at C and D , one of which would be zero.

Critical Points.—It appears from the foregoing discussion that, when the load-series consists wholly of concentrated loads, the stress can have a maximum or a minimum value only when a load is passing some one of the four points A, B, C and D . This might be inferred also from Equation (3), since only in exceptional cases can this equation be satisfied without there being a load at one of these points. Such a load may, by a slight displacement, be divided in any desired ratio between the adjacent segments of the span.

These four points may be called critical points, and may be distinguished as "maximum points" and "minimum points," according as the passage of a load through them is associated with a maximum or a minimum value of the stress.

The foregoing discussion shows that, at a maximum point, the influence line is convex upward (*i. e.*, toward the positive side), while at a minimum point it is concave upward. In applying this statement to the points A and B , the influence line is to be extended beyond the span as a horizontal line.

When only a single one of the four critical points is occupied by a load, or if such occupied points are either all maximum points or all minimum points, the form of the influence line thus shows at a glance whether the stress is a maximum or a minimum. To make the discussion complete it is necessary to consider cases in which Equation (3) is satisfied while loads are passing both maximum and minimum points.

General Criterion.—In order to deduce a completely general rule for discriminating between maximum and minimum values, consider Equation (2) deduced above:

$$\frac{dM}{dx} = \frac{l_1}{l} (P_2 + \frac{n_2}{n} Q) - \frac{l_2}{l} (P_1 + \frac{n_1}{n} Q) \dots \dots \dots (2)$$

This value of $\frac{dM}{dx}$ remains constant, except when the load (either concentrated or distributed), passes some one or more of the "critical" points A, B, C or D , thus changing the values of one or more of the quantities P_1, P_2 or Q .

Assuming a small displacement of the load-series, let loads W_a , W_b , W_c and W_d , respectively, pass the four critical points A , B , C and D . The change in the value of $\frac{dM}{dx}$ may be found from the changes in P_1 , P_2 and Q . For a positive displacement (the loads moving toward the left),

$$\begin{array}{lll} P_1 \text{ changes by the amount } & W_c - W_a; \\ P_2 & & W_b - W_d; \\ Q & & W_d - W_c. \end{array}$$

Hence the change (written as an increase) in the value of $\frac{dM}{dx}$ is

$$\delta \left(\frac{dM}{dx} \right) = \frac{l_1}{l} [W_b - W_d + \frac{n_2}{n} (W_d - W_c)] - \frac{l_2}{l} [W_c - W_a + \frac{n_1}{n} (W_d - W_c)]; \text{ which reduces to}$$

$$\delta \left(\frac{dM}{dx} \right) = \frac{l_2}{l} W_a + \frac{l_1}{l} W_b - \frac{n_2}{n} W_c - \frac{n_1}{n} W_d \dots \dots (4)$$

According as this value is positive or negative, the value of $\frac{dM}{dx}$ increases or decreases as the loads pass through the supposed position.

As a position of maximum stress is passed, $\frac{dM}{dx}$ decreases; as a position of minimum stress is passed, $\frac{dM}{dx}$ increases.

This is the general criterion for discriminating between maximum and minimum values of the stress. The application to special cases, in which l_1 , l_2 , n_1 and n_2 are known in magnitude and sign, presents no difficulty.

The conclusions above reached from the study of the influence diagrams for the six cases, may easily be verified by inspection of Equation (4). Thus, if l_1 , l_2 , n_1 and n_2 are all positive (Fig. 1), the first and second terms in the value of $\delta \left(\frac{dM}{dx} \right)$ are intrinsically positive, while the third and fourth are negative. Therefore, for a maximum stress either W_c or W_d must be present, while for a minimum stress either W_a or W_b must be present. In the case shown in Fig. 2, the third term in the value of $\delta \left(\frac{dM}{dx} \right)$ drops out, n_2 being zero; the conclusions are otherwise the same as in the preceding case.

In Fig. 3, l_1 and n_1 being negative, the "A" and "D" terms in the value of $\delta \left(\frac{dM}{dx} \right)$ are intrinsically positive, while the "B" and "C"

terms are negative. Hence A and D are minimum points, while B and C are maximum points. The same conclusions apply to the case shown in Fig. 4, although every term in the second member of Equation (4) is infinite.

In Fig. 5, n_1 is negative, while l_1 , l_2 and n_2 are positive. Therefore, in Equation (4) the " A ," " B " and " D " terms are positive, while the " C " term is negative. Hence C is the only maximum point.

In Fig. 6, n_2 being negative, the only negative term in Equation (4) is the " D " term. Hence D is the only maximum point.

If all but one of the four terms in the value of $\delta \left(\frac{dM}{dx} \right)$ are zero when the condition for maximum or minimum stress (Equation 3) is satisfied, the discrimination between maximum and minimum values is thus very simple, being readily accomplished, either by simple inspection of the influence diagram, or by noticing the sign of the single term in the second member of Equation (4). If two or more terms remain, Equation (4) must be applied.

Distributed Loads.—Consider next the case in which there is a distributed load at each of the four points A , B , C and D . Equation (4) is still applicable, but its form may conveniently be changed. In accordance with the notation used in the deduction of Equation (1), let w denote the load per unit length at any point, and let w_a , w_b , w_c and w_d be the values of w at A , B , C and D . If the loads receive a positive displacement δx , we must put, in Equation (4):

$$W_a = w_a \delta x, W_b = w_b \delta x, W_c = w_c \delta x \text{ and } W_d = w_d \delta x.$$

The result is:

$$\delta \left(\frac{dM}{dx} \right) = \left(\frac{l_2}{l} w_a + \frac{l_1}{l} w_b - \frac{n_2}{n} w_c - \frac{n_1}{n} w_d \right) \delta x \dots (4a)$$

$$\text{or} \quad \frac{d^2 M}{dx^2} = \frac{l_2}{l} w_a + \frac{l_1}{l} w_b - \frac{n_2}{n} w_c - \frac{n_1}{n} w_d \dots \dots \dots (5)$$

A position of the loads satisfying Equation (3) gives a maximum or a minimum value of the stress, according as the value of $\frac{d^2 M}{dx^2}$ is negative or positive. If the value of w is known at every point, the application of Equation (5), to any particular case in which l_1 , l_2 , n_1 and n_2 are known, presents no difficulty.

Combination of Concentrated and Distributed Loads.—Another case is that in which there are concentrated loads at certain of the critical

points, while at others the load is distributed. This occurs when the load consists of one or more locomotives followed by a uniformly distributed train-load, as often prescribed in bridge specifications. The value of $\delta \left(\frac{dM}{dx} \right)$ will then contain part of the terms in Equation (4) and part of those in Equation (4a). Since the latter terms contain the factor δx , they will be very small in comparison with the former, and may, therefore, generally be disregarded in applying the equation.

DETERMINATION OF GREATEST MAXIMUM.

In those cases in which loads in all positions cause the same kind of stress in the member under consideration, it is evident that for greatest stress the span should be loaded as fully as possible, and that the heaviest loads should be so placed as to produce as great an effect as possible. Thus in Figs. 1, 2 and 6 the heavy loads should be near *D*; in Fig. 5, near *C*. This is, of course, only a rough, general rule.

When a reversal of stress is possible, as in Figs. 3 and 4, the two kinds of stress must be considered separately. Thus, for one kind of stress, loads should cover only the portion *AE*; for the other kind, only the portion *EB*. These rules are not absolute; it may be, that, if loads are carried beyond *E*, the diminution of stress due to such loads will be more than counterbalanced by an increase due to bringing other loads nearer to their positions of greatest effect.

By inspection of the influence diagram it is thus possible to determine, with great probability in many cases, which of two maximum values is greater, without actually computing the values. In some cases, however, it may be necessary to compute the stress for each of two or more positions, in order to determine with certainty which maximum value is the greatest.

GENERAL RESULTS.

The determination of the position of a series of moving loads which will cause the greatest stress in any member of a truss thus involves the following steps:

(1) The approximate determination of the required position by inspection of the influence diagram.

(2) The testing of the exact criterion expressed by Equation (3), or one of the equivalent Equations (3a), (3b), (3c).

(3) The application of Equation (4) or Equation (5) to determine with certainty whether the stress is a maximum or a minimum. In the majority of practical cases this is, however, unnecessary, the inspection of the influence diagram being sufficient.

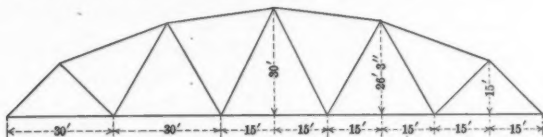


FIG. 8.

NUMERICAL APPLICATIONS.

In order to illustrate clearly the application of the general criterion to different cases, numerical examples will now be solved, involving the cases shown in Figs. 1, 3, 4 and 5.

The load-series will be assumed to consist of two locomotives followed by a uniformly distributed load, the loads and intervals being given in Fig. 9. In practice, a diagram should be drawn, showing the lines of action properly spaced, and a skeleton diagram of the truss

10	35	60	85	110	125	140	155	170	180	205	230	255	280	295	310	325	340	Total Load (thou's of lbs.)
10	25	25	25	25	15	15	15	15	10	25	25	25	25	15	15	15	15	Loads (thou's of lbs.)
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	
○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	2.5 per foot
7.5'	5'	5'	5'	5'	9'	5'	6'	5'	8'	7.5'	5'	5'	5'	9'	5'	6'	5'	5'
7.5'	12.5'	17.5'	22.5'		31.5'	36.5'	42.5'	47.5'	55.5'	63.0'	68.0'	73.0'	78.0'	87.0'	92.0'	98.0'	103.0'	Dist. from 1st. Load

FIG. 9.

prepared on a movable strip of paper or on tracing cloth, to facilitate the trial of different positions.

(1) *Member of Loaded Chord.*—Consider, first, the case shown in Fig. 1, the dimensions of the truss being as marked on Fig. 8. For greatest stress in the member k the truss must be fully loaded, and the effect of the heaviest loads will be greatest if these are near the point D . A maximum value can occur only with a load at C or at D . It remains to test the general equation (3) for such positions as these general considerations suggest.

From the dimensions of the truss, we have $l = 150$ ft., $l_1 = 45$ ft.,

$l_2 = 105$ ft., $n = 30$ ft., $n_1 = 15$ ft. and $n_2 = 15$ ft. Equation (3a) therefore becomes $P_1 + \frac{1}{2} Q = \frac{3}{10} W$.

It is seen that there is a conflict between the requirements that the span shall be fully loaded, and that the heaviest loads shall be near D .

It is found that there are two positions in which the conditions for maximum stress are satisfied—with either the fourth or the fifth load at C . Between two maxima there must be a minimum; and it is found that between the two positions of maximum stress there is another position in which Equation (3a) is satisfied. The numerical data for testing these three positions and two others are shown in Table No. 1.

TABLE No. 1.

Position.	W	$\frac{l_1}{l} W$	P_1	Q	$P_1 + \frac{n_1}{n} Q$	
Third load at $C..$	401 250	120 375	35 000 60 000	105 000 120 000 80 000 95 000	87 500 95 000 100 000 107 500	Criterion not satisfied.
Fourth load at C	413 750	124 125	60 000 85 000	95 000 110 000 70 000 85 000	107 500 115 000 120 000 127 500	Criterion satisfied. Maximum.
Fourth load 4.5 ft. left of $C....$	425 000	127 500	85 000	85 000	127 000	Criterion satisfied. Minimum.
Fifth load at $C...$	436 250	127 875	85 000 110 000	85 000 60 000	127 500 140 000	Criterion satisfied. Maximum.
Tenth load at $D..$	433 750	130 125	110 000	60 000 70 000	140 000 145 000	Criterion not satisfied.

It happens that when the fourth load is at C , the ninth is at D . Hence, Q may include any arbitrary portion of each of these two loads. Thus, if the whole of the fourth load is included in Q , the value of P_1 is 60 000 lbs., while Q may be either 95 000 lbs. or 110 000 lbs. or anything between, depending upon what portion of the ninth load is included in Q . If the whole of the fourth load is included in P_1 , the value of P_1 is 85 000 lbs., while Q has any value from 70 000 to 85 000 lbs. The values corresponding to these four limiting cases are shown in Table No. 1. Evidently $P_1 + \frac{n_1}{n} Q$ may have any value between 107 500 lbs. and 127 500 lbs.; and since $\frac{l_1}{l} W = 124 125$ lbs., the Equation $P_1 + \frac{n_1}{n} Q = \frac{l_1}{l} W$ is satisfied.

Since, in the position of minimum stress the point C is 4.5 ft. from the fourth load and only 0.5 ft. from the fifth, it may be inferred that the stress is greater when the fourth load is at C than when the fifth is there. This may be verified by computing the values of the stress.

The general criterion is satisfied in no other position which seems favorable for greatest stress. The conclusion is that the member has its greatest stress when the fourth load is at C .

(2) *Web Member*.—Consider next the case of a web member, shown in Fig. 3, the dimensions of the truss being as marked in Fig. 8. The influence line shows that the stress is subject to reversal, so that tension and compression must be considered separately. It will suffice to consider the case of compression, corresponding to negative values of the stress-moment.

Inspection of the influence line shows that the load should cover the portion EB of the span; that a load has its greatest effect when at D ; and that a maximum compression can occur only when a load is at D (the case of a maximum with a load at A being excluded because the portion of the span AE should be free from loads). It remains to apply the exact criterion.

A trial of successive positions shows that the criterion for maximum stress is satisfied when the second load is at D , but not in any other position which seems favorable for greatest stress. The conclusion is that this position gives the greatest compression in the member. The data for testing three positions are shown in Table No. 2.

From the dimensions of the truss, as shown in Fig. 8, we have $l = 150$ ft., $l_1 = -25$ ft., $l_2 = 175$ ft., $n = 30$ ft., $n_1 = -55$ ft., $n_2 = 85$ ft., $\frac{l_1}{l} = -\frac{1}{6}$ and $\frac{n_1}{n} = -\frac{11}{6}$.

TABLE No. 2.

Position.	W	$\frac{l_1}{l} W$	P_1	Q	$P_1 + \frac{n_1}{n} Q$	
First load at D .	295 000	-49 167	0	0 10 000	0 - 18 333	Criterion not satisfied.
Second load at D .	310 000	-51 667	0	10 000 35 000	- 18 333 - 64 167	Criterion satisfied. Maximum.
Third load at D .	325 000	-54 167	0	35 000 60 000	- 64 167 -110 000	Criterion not satisfied.

(3) *Web Member in a Truss with Parallel Chords.*—In Fig. 4, a negative stress-moment corresponds to tension in the member k . For greatest tension in this member, the influence line shows that only the portion EB of the span should be loaded, and that a load must be at D .

Let the span consist of eight panels of 30 ft. each. Then $l = 240$ ft., $l_1 = -\alpha$, $l_2 = +\alpha$, $n = 30$ ft., $n_1 = -\alpha$, $n_2 = +\alpha$, $\frac{l_1}{n_1} = 1$, and $\frac{l_2}{n_2} = 1$. The criterion for maximum or minimum stress reduces to Equation (3c), that is,

$$Q = \frac{n}{l} W = \frac{1}{8} W.$$

It is found that this equation is satisfied when either the third or the fourth load is at D . Both these positions give maximum values of the stress. Between them is a position of minimum stress, which, by applying Equation (3c), is found to be that in which the third load is 1.5 ft. to the left of D . Comparing this with the two positions of maximum stress, it seems highly probable that the greater of the two maximum values is that which occurs when the fourth load is at D . The numerical data for these three positions are shown in Table No. 3.

TABLE No. 3.

Position.	W	$\frac{n}{l} W$	Q	
Third load at D	476 250	59 530	35 000 60 000	Criterion satisfied. Maximum.
Third load 1.5 ft. left of D ..	480 000	60 000	60 000	Criterion satisfied. Minimum.
Fourth load at D	488 750	61 094	60 000 85 000	Criterion satisfied. Maximum.

(4) *Chord Member, Truss with Subordinate Bracing.*—The case represented in Fig. 5 is peculiar in that l_1 and n_1 have opposite signs. The form of the influence diagram shows that for greatest stress the span should be fully loaded, and that a load must be at C for a maximum stress.

Let the span consist of five main panels of 60 ft. each, subdivided into ten panels of 30 ft. each; then $l = 300$ ft., $l_1 = 60$ ft., $l_2 = 240$ ft., $n = 30$ ft., $n_1 = -30$ ft. and $n_2 = 60$ ft. Equation (3a) reduces to $P_1 - Q = \frac{1}{5} W$.

Trying successively the positions which appear favorable for greatest stress, it is found that the criterion is satisfied when the thirteenth load is at *C*, but not with either the twelfth or the fourteenth at *C*. This is shown in Table No. 4.

TABLE No. 4.

Position.	<i>W</i>	$\frac{1}{5} W$	<i>P</i> ₁	<i>Q</i>	<i>P</i> ₁ - <i>Q</i>	
Twelfth load at <i>C</i> ..	765 000	153 000	205 000 230 000	120 000 106 000 95 000 80 000	85 000 100 000 135 000 150 000	Criterion not satisfied.
Thirteenth load at <i>C</i>	777 500	155 500	230 000 255 000	110 000 95 000 85 000 70 000	120 000 135 000 170 000 185 000	
Fourteenth load at <i>C</i>	790 000	158 000	255 000 280 000	85 000 60 000	170 000 220 000	

When the thirteenth load is at *C*, there is also a load at *D*. Since *D* is a "minimum-point," it is strictly necessary to apply Equation (4) to determine with certainty whether the stress is a maximum or a minimum. We have $W_c = 25\,000$ lbs., $W_d = 15\,000$ lbs., $\frac{n_2}{n} = 2$, and $\frac{n_1}{n} = -1$; and Equation (4) becomes

$$\delta \left(\frac{dM}{dx} \right) = -50\,000 + 15\,000 = -35\,000.$$

Since this value is negative, while *M* is positive, the stress is a maximum.

The conclusion is that the greatest stress occurs when the thirteenth load is at *C*.

DISCUSSION.

Mr. Cilley. F. H. CILLEY, Esq. (by letter).—The author, in his exposition of criteria for the position of loads causing maximum stress, offers an interesting extension of the well-known rule for loading for maximum moment at a given section of a beam, that the average load on both sides of the section shall be equal. His principle applies to all cases in which the influence line consists of three lines, which include most cases arising in connection with statically determined frameworks. And, in the case of an influence line whose end lines intersect in the region of the middle line, the rule still possesses a simple objective interpretation like that of the earlier moment rule, which applied only to two-line influence lines.

It may, perhaps, be of interest to note some very simple and wholly general facts in connection with influence lines. First, the influence line for the stress in any bar of any framework whatsoever, determinate or indeterminate, consists simply of a series of straight lines, if we neglect secondary stresses. It follows, therefore, that the variation in

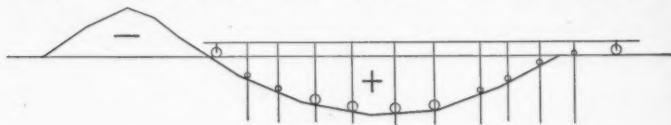


FIG. 10.

stress in any such bar, due to the movement along the load line of a series of concentrated loads maintained at invariable distances apart, is a linear function of the amount of movement of the load. A linear function of a variable can become a maximum or minimum only at a point of discontinuity of the function, that is to say, in this case, only when some load passes an angle of the influence line. For a maximum, it is evident that the angle must be concave toward the zero line in a region of positive influence, convex toward the zero line in a region of negative influence; and for a minimum, the reverse is the case.

A particularly objective and interesting general rule as to the position of loads for maximum influence is found in the following:

Imagine (in Fig. 10), a track of the form of the influence line with positive part down and zero line horizontal. Imagine a series of small wheels, the weights of which are proportioned as those of an actual series of loads, and which are free to move vertically, independently; but horizontally only all together, that is, with constant horizontal spacing. A position of stable equilibrium of this series of wheels on the influence-line track will correspond to a position for maximum influence of the corresponding actual loads with similar spacing. Here,

it is very evident that some load must be at an angle concave from Mr. Cilley above. The reasons for this rule will readily be perceived. Its immediate source, if the writer's memory is not in error, is an Italian work on graphic statics.

F. C. KUNZ, M. Am. Soc. C. E. (by letter).—The method of influence lines is not so often used as other methods, although it is the simplest way to determine stresses, even for uniform loading, in many cases, and particularly for web-members of all but Pratt trusses, bending moments of pins when simultaneous stresses have to be found, etc.

On page 250 the author gives a general rule for drawing influence lines. For use in special cases, all similar methods derived from the theory of moments offer for web-members but little advantage, as the point of intersection of the top and bottom chords, lies, as a rule, beyond the drawing board. Of all the methods published on this subject, the writer gives in the following the simplest he has yet found. It was first published by Professor R. Land in Constantinople,* and applies to all cases of trusses with verticals and at least one horizontal chord.

For a diagonal C_1 D rising to the left,

Fig. 11 (and $C D_1$ rising to the right, Fig. 12):

Produce the top chord of the panel in question to its intersection A_1 , (B_1), with the end vertical A , (B), and join A_1 and B_1 (B_1 and A); then the area $A_1 C_1 D B A_1$, (area $A C D_1 B_1 A$), is the influence area when $\alpha =$ unit load. Its + (tension) and - (compression) value is found by simple inspection.

For a vertical $D D_1$, Fig. 11 (and Fig. 12):

Produce the top chord next to the vertical to its intersection A_1 , (B_1), as before; then the area $A_1 D_1 E B A_1$, (area $A C D_1 B_1 A$) is the influence area when $h =$ unit load.

Professor Land derives these rules in a general way by means of

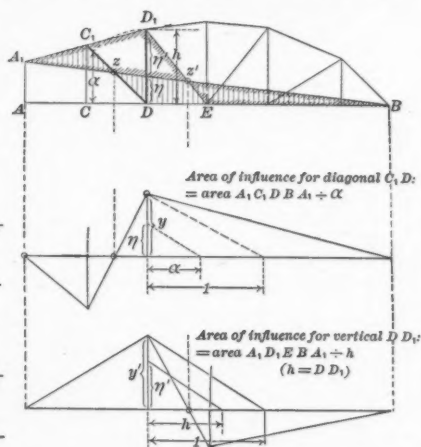


FIG. 11.

* Centralblatt der Bauverwaltung, 1897.

Mr. Kunz. cinemactical principles. The writer gives, in the following, a proof, using the general rule on page 250.

Given four verticals F , A , C and D and a point B (Fig. 13). From B draw any two lines, join $F_1 A_1$, produce this to C_1 , join $C_1 D_1$; then the locus of the point of intersection of $C_1 D_1$ and $A_1 B$ is the same vertical z for any two lines drawn from B . Since (Fig.

$$14), \frac{m_1}{m_2} = \frac{y_1}{y_2}, \text{ and } y_1 = \frac{a m}{l_1}, \text{ and } y_2 =$$

$$\frac{n_3}{l_2}; \text{ hence } \frac{m_1}{m_2} =$$

$$\frac{m l_2}{n_3 l_1}, \text{ the position of}$$

the z -vertical depends only on the relative position of the verticals F , A , C and D and the point B .

In drawing the two lines from B through the skeleton of the truss in such a way that BA corresponds to the one and BA_1 to the other, we see directly that as z_1 lies vertically above z it represents the zero point, and hence the area $A_1 C_1 D B A_1$ in the skeleton is the influence area of the stress in the diagonal $C_1 D$.

There remains only to find its scale. In other words, by what unit load α must η_1 be divided to give

$$y_1, \text{ i. e., } \frac{\eta_1}{\alpha} = y_1 \text{ and } \alpha = \frac{\eta_1}{y_1}$$

$$y_1 = \frac{a m}{l_1}, \text{ substituting for } \alpha \text{ its value (see page 250) } \alpha = \frac{l_1 l_2}{l t},$$

we have $y_1 = \frac{l_2 m}{l t}$. As $\eta_1 = h_1 - \frac{h_1 l_1}{n_1} \frac{n_4}{l}$ we get, after a short transformation, $\alpha = \frac{\eta_1}{y_1} = \frac{h_1 t}{n_1}$. Further, $\triangle F G D \sim \triangle C C_1 D$,

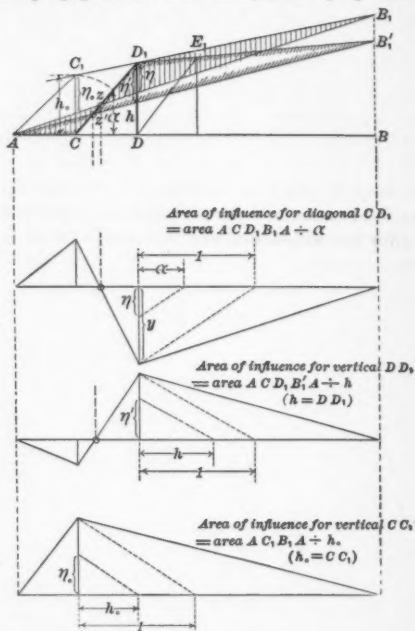


FIG. 12.

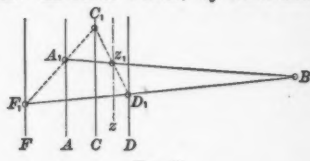


FIG. 13.

hence $t = \frac{h n_2}{d}$, where d denotes

$$\overline{C_1 D}. \text{ Thus we get } \alpha = \frac{h_1^2 n_2}{d n_1},$$

and as $\frac{h_1 n_2}{n_1} = h_2$, we get

the unit load $\alpha = \frac{h_1 h_2}{d}$, which

can easily be scaled off in the skeleton. From the proportion

$$\frac{y_1}{1} = \frac{\eta_1}{\alpha} \text{ follows the}$$

construction of the influence line to any desired scale.

For the vertical $D D_1$ (Fig. 15) it follows that the point z_1 is the zero point of the influence area $A_1 D_1 E B A_1$. Using the notation in Fig.

15, we have again $y_1 = \frac{a m}{l_1}$,

where $a = \frac{l_1 l_2}{l t}$, and $t = l_1$

+ m ; $\eta_1 = h - \frac{h l_1 n_4}{l n_1}$. After

a short transformation

we get $\frac{\eta_1}{y_1} =$

$h = \text{unit load, and}$

$$\frac{y_1}{1} = \frac{\eta_1}{h}.$$

Fig. 16 shows the application of the above rule for the diagonals of a half truss of a drawbridge considered as a single span, which will be of particular use in determining the bending moment on the pin D for this condition.

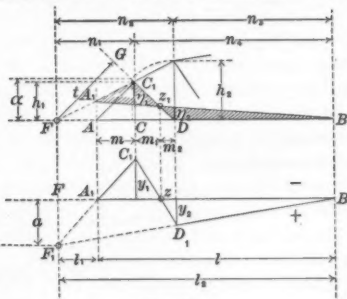


FIG. 14.

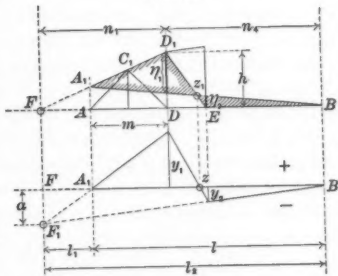


FIG. 15.

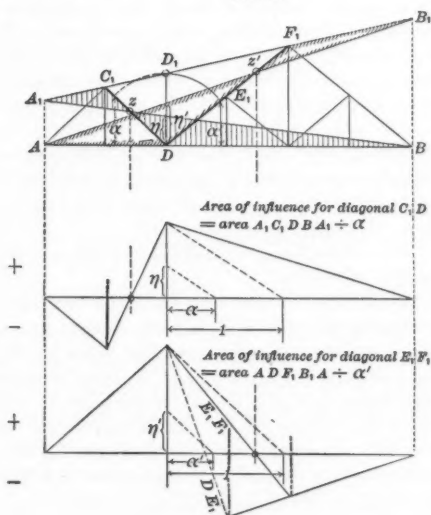


FIG. 16.

Mr. Kunz.

Mr. Hoskins. L. M. Hoskins, Esq. (by letter).—It was no part of the object of this paper to advocate the use of wheel concentrations as a basis of bridge design. This kind of specification has been in general use for a considerable number of years, and, doubtless, will continue to be used for a long time to come, in spite of the efforts of its opponents to supersede it. Even if some different specification should be adopted generally, many engineers of railroad companies probably will think it advisable to apply the wheel-concentration method in estimating the effect of new and heavier train-loads on existing bridges. In view of these facts it seemed worth while to present what appeared to the author to be a material simplification in the treatment of the general problem of computing stresses due to any series of loads whatever.

The members who participated in the discussion at the last Annual Convention of the Society were unanimous in condemning the method of wheel concentrations. They were, however, by no means agreed as to what method should replace it. Thus, the method advocated by Mr. Morison was condemned by Mr. Waddell as complicating matters worse than ever. And, while the majority appeared to favor some form of "equivalent uniform-load," each had his own opinion as to how this should be estimated. That this discussion does not represent the unanimous opinion of those engineers most directly interested in the question is indicated by recent correspondence in *Engineering News*.*

No one supposes that the method of wheel concentrations can represent with exactness the actual distribution of loads upon a bridge. Its advocates merely contend that it comes a step nearer than other methods to representing actual conditions. The truth of this contention is tacitly admitted by most of those who condemn the method. The correctness of every proposed method is tested by a comparison of its results with those of the wheel-load method.

The only real objection to the wheel-concentration method, as compared with other proposed methods, is the difficulty of the computations involved. How serious this objection is can be judged only by one who is thoroughly familiar with the best methods of computation. Wheel concentrations appear to present no serious difficulties to the computer who familiarizes himself with the best methods of dealing with them.

Aside from whatever practical value it has, the problem discussed in the paper possesses considerable mathematical interest. The properties of influence lines given by Mr. Cilley and Mr. Kunz are interesting contributions to the discussion. With reference to the comment of Mr. Kunz, on the general rule for drawing influence diagrams, it may be remarked that when the moment-center falls beyond convenient limits, the ordinates at *C* and *D* can be computed readily by the general values given on page 250.

* July 18th and September 7th, 1899.